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# Delta-Type Dirac Point Interactions and Their Nonrelativistic Limits

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The problem of a relativistic free particle on a line with a hole, which is characterized in terms of boundary conditions for a one-dimensional Dirac Hamiltonian perturbed at one point, is reviewed. We show that the general four-parameter family of point interactions earlier obtained by Falkensteiner and Grosse can be written in two forms: In one of them three subfamilies of boundary conditions are obtained. In the nonrelativistic limit one of these subfamilies coincides with those given by Carreau *et al.* and Carreau. In the other form, three subfamilies of boundary conditions are also obtained, two of which coincide with those studied by Benvegnù and Dabrowski. In the nonrelativistic limit all these subfamilies coincide with those studied by Albeverio *et al.* The most general subfamilies for which the Dirac Hamiltonian is invariant under space inversion $\Pi$ as well as under time reversal $T$ and $\Pi T$ are obtained. Only these subfamilies represent delta-type Dirac point interactions. Typical relativistic and nonrelativistic boundary conditions are therein included.

## 1. INTRODUCTION

The point interactions in one space dimension may be used to approximate in a simple way more structured and complex physical situations. The simplest nonrelativistic point interaction in one-dimensional quantum mechanics was introduced by Fermi [1] and corresponds to a Dirac delta. Its mathematical interpretation was given by Berezin and Faddeev [2]. In one-dimensional relativistic quantum mechanics, the free Dirac operator perturbed by point interactions was first studied by Woods and Callaway [3] and corresponds to the relativistic generalization of the classical Kronig–Penney model [4].

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Subsequently, interest in the relativistic Kronig–Penney model increased [5]. At the same time, it was noted that the method of obtaining the band equation in the Kronig–Penney model in the delta function limit, and the method of directly solving the Dirac equation for a delta function potential, lead to different results. As noted by Subramanian and Bhagwat, this peculiarity can be seen even when the potential consists of a single delta function term. However, it was noted that these results became identical when the strength of the delta function potential was weak. Fairbairn *et al.* [6] argued that one should require that the strength of the potential be very small. In that case the relativistic corrections may be small, but since they are precisely the terms of interest, they should be treated properly. Steps leading to the erroneous use of the usual treatment of the delta function potential in the one-dimensional Dirac equation have been explained [7].

It is known that there is a general four-parameter family of self-adjoint Dirac free massless operators describing point interactions [8]. For a free Dirac operator with mass, the same four-parameter family of boundary conditions is obtained for the Dirac Hamiltonian perturbed at one point as well as for the Dirac Hamiltonian for a particle in a one-dimensional box [9]. The physical literature deals with particular one-parameter subfamilies [10]. In order to understand and solve the ambiguities and controversy [11] with the relativistic point interactions, the resolvent, spectrum, and scattering matrix were calculated [12]. However in ref. 12 a four-parameter family of self-adjoint extensions different from that given in ref. 8 was obtained. Also, in ref. 12 their nonrelativistic limits are shown to coincide with the four-parameter family of self-adjoint Schrödinger operators studied in ref. 13. The full nonrelativistic families were studied in ref. 14. Recently, these nonrelativistic boundary conditions were studied with a detailed description of fundamental symmetry transformations: parity, time reversal, and scaling [15].

In recent years, various aspects of relativistic point interactions have been considered. The question of approximation by smooth potentials was studied [16], as well as the effect of the so-called “Coulomb potential” on the Dirac Hamiltonian [17] (relativistic point interactions due to a potential singular at one point). The delta potentials also have been applied to relativistic particle physics [18]. There also have been attempts to generalize the relativistic point interaction model to higher dimensions (such a generalization is possible in the Schrödinger case [19]). They lead the theorem of Svendsen [20], which claims that a nontrivial relativistic point interaction can only be defined for a dimension equal to one. Recently, using a path-integral representation of the one-dimensional Dirac particle with point interaction, the corresponding Green function by means of a perturbation expansion has been obtained [21].

As is well known, discrete symmetries play an important role in quantum mechanics and even more in relativistic quantum mechanics. In fact, they are capable of restricting families of boundary conditions (self-adjoint extensions) and selecting some of them. The relativistic time-reversal symmetry was briefly mentioned in a recent paper [22]. It is claimed therein that when time-reversal invariance is imposed on the four-parameter family of point interactions for the Dirac equation, the number of parameters is reduced to three. However, the various discrete symmetries of the one-dimensional Dirac Hamiltonian perturbed at one point, as far as we know, have not been considered.

In this paper, we use the full four-parameter family of self-adjoint extensions (boundary conditions) of the Dirac Hamiltonian operator with point interactions obtained in ref. 8, but in the Dirac representation. This family can be written as three types of subfamilies of self-adjoint extensions (see Appendix A). In the nonrelativistic limit one of these subfamilies coincides with those given by Carreau *et al.* [23] and Carreau [24] (see Appendix C). Likewise, the full four-parameter family of boundary conditions of [8] can be again written as three different types of subfamilies (see Appendix B), two of which coincide with those studied by Benvegnù and Dabrowski [12]. In the nonrelativistic limit all these subfamilies coincide with those studied by Kurasov [14] and Albeverio *et al.* [15]. In order to relate and unify in a simple way the results of the above-mentioned authors, some overlap was necessary.

In Section 2, we present the family of self-adjoint extensions of the Dirac Hamiltonian operator with point interactions. In Section 3, the subfamilies of boundary conditions for which the Dirac Hamiltonian is invariant under time reversal $T$, $\Pi T$, and space inversion $\Pi$ are obtained. Only these boundary conditions may represent delta-type relativistic point interactions. Typical relativistic delta-like boundary conditions as well as their nonrelativistic limits are obtained. In particular, the usual Dirac delta relativistic point interaction is included; this is reviewed in Section 4. Finally, the conclusions are presented in Section 5.

## 2. SELF-ADJOINT EXTENSIONS OF THE HAMILTONIAN OPERATOR

For a relativistic free particle [i.e., $V(x) = 0$] moving on a line with the origin excluded, the Dirac equation may be written as

$$i\hbar \frac{\partial}{\partial t} \Psi(x, t) = \left(-i\hbar c\alpha \frac{\partial}{\partial x} + mc^2\beta\right)\Psi(x, t) \tag{1}$$

where $\Psi$ denotes a two-component wavefunction ("spinor") depending on

$x \in \Omega = \mathcal{R} - \{0\}$ and time. The $2 \times 2$ matrices $\alpha$ and $\beta$ satisfy $\alpha\beta + \beta\alpha = 0$ and $\alpha^2 = \beta^2 = 1$. In the Dirac representation $\alpha = \sigma_x$ and $\beta = \sigma_z$.

The Dirac eigenvalue equation is given by

$$H\psi(x) = \left(-i\hbar c\alpha \frac{d}{dx} + mc^2\beta\right)\psi(x) = E\psi(x) \tag{2}$$

where $\psi$ is related to $\Psi$ by $\Psi(x, t) = \psi(x)e^{-iE_t/\hbar}$. In the Dirac representation we write $\psi = \binom{\phi}{\chi}$, where $\phi$ and $\chi$ are, respectively, the spatial parts of the so-called large and small components of the Dirac spinor. The "spinors" $\psi(x)$ and $(H\psi)(x)$ belong to a dense proper subset of the Hilbert space $\mathsf{H} = \mathsf{L}^2(\Omega) \oplus \mathsf{L}^2(\Omega)$.

It is found that there is a four-parameter family of self-adjoint Hamiltonian (extensions) that can be characterized by a four-parameter family of boundary conditions imposed on the components of the Dirac "spinor" [8]. In the Dirac representation, the formal Hamiltonian $H \equiv H_{\theta,\mu,\tau,\gamma}$ is

$$H_{\theta,\mu,\tau,\gamma} = -i\hbar c\sigma_x \frac{d}{dx} + mc^2\sigma_z \tag{3}$$

whose domain is given by

$$Dom(H) = \left\{\psi = \begin{pmatrix}\phi\\ \chi\end{pmatrix} \middle| \psi \in \mathsf{H}, \quad \text{a.c.} \quad \text{in} \quad \Omega, \quad (H\psi) \in \mathsf{H}, \quad \psi \text{ fulfils} \right.$$
$$\left. \begin{pmatrix}\phi(0+) + \chi(0+)\\ \phi(0-) - \chi(0-)\end{pmatrix} = U\begin{pmatrix}\phi(0+) - \chi(0+)\\ \phi(0-) + \chi(0-)\end{pmatrix}, U^{-1} = U^{+}\right\} \tag{4}$$

where hereafter a.c. means absolutely continuous function and the superscript plus sign denotes the adjoint of a vector or a matrix. The unitary matrix $U$ may be written as

$$U = \begin{pmatrix}a & b\\ c & d\end{pmatrix} \tag{5}$$

where $a$, $b$, $c$, $d$ imply the conditions

$$a\bar{a} + b\bar{b} = c\bar{c} + d\bar{d} = b\bar{b} + d\bar{d} = a\bar{a} + c\bar{c} = 1 \tag{6}$$
$$a\bar{c} + b\bar{d} = a\bar{b} + c\bar{d} = 0$$

which are satisfied with $a = e^{i\mu}\, e^{i\tau} \cos\theta$, $b = e^{i\mu}\, e^{i\gamma} \sin\theta$, $c = e^{i\mu}\, e^{-i\gamma} \sin\theta$, and $d = -e^{i\mu}\, e^{-i\tau} \cos\theta$, where $0 \le \theta < \pi$ and $0 \le \mu, \tau, \gamma < 2\pi$. It is worth noting that these parameters take only finite values, though the same boundary condition may be given by a subfamily of parameters.

## 3. Π, T, AND Π*T* SYMMETRIES

Our aim is to characterize the most general delta-type relativistic point interactions. That is, among all the boundary conditions included in *Dom*($H$), we want to single out those for which the Dirac Hamiltonian perturbed at one point is invariant under space reflection Π, which is a natural requirement for a δ-type interaction, and then we shall consider time-reversal $T$ and $\Pi T$ invariance.

Let us consider these transformations in the Dirac representation:

$$\begin{aligned} \Pi\Psi(x, t) &= \sigma_z\Psi(-x, t) \\ T\Psi(x, t) &= \sigma_z\overline{\Psi(x, -t)} \\ \Pi T\Psi(x, t) &= \overline{\Psi(-x, -t)} \end{aligned} \tag{7}$$

where $\overline{\Psi}$ is the complex conjugate of Ψ.

Let us require that the Hamiltonian operator with domain given in (4) be invariant under the parity transformation Π; then

$$\Pi^{-1} H\Pi\psi = H\psi \tag{8}$$

Clearly, the parity-transformed "spinor" must satisfy $\Pi\psi \in Dom(H)$, which implies that

$$U\sigma_x = \sigma_x U \tag{9}$$

Thus, the parameters $a$, $b$, $c$, $d$ in $U$ satisfy

$$a = d, \qquad b = c \tag{10}$$

These relations yield four subfamilies of two-parameter unitary matrices:

$$U = e^{i\mu}\begin{pmatrix} \pm i\cos\theta & \sin\theta \\ \sin\theta & \pm i\cos\theta \end{pmatrix} \tag{11}$$

where the upper sign corresponds to $\gamma = 0$, $\tau = \pi/2$ and the lower one to $\gamma = 0$, $\tau = 3\pi/2$, and

$$U = e^{i\mu}\begin{pmatrix} \pm i\cos\theta & -\sin\theta \\ -\sin\theta & \pm i\cos\theta \end{pmatrix} \tag{12}$$

where the upper sign corresponds to $\gamma = \pi$, $\tau = \pi/2$ and the lower one to $\gamma = \pi$, $\tau = 3\pi/2$.

If the Hamiltonian operator is invariant under time reversal $T$ we have

$$T^{-1} HT\psi = H\psi \tag{13}$$

For that, the "spinor" ψ must additionally satisfy $T\psi \in Dom(H)$. So, the matrix $U$ satisfies

$$U^{+} = \overline{U} \tag{14}$$

which implies that

$$b = c \tag{15}$$

Therefore, we are left with two subfamilies of unitary matrices with three real, independent parameters:

$$U = e^{i\mu}\begin{pmatrix} e^{i\tau}\cos\theta & \pm\sin\theta \\ \pm\sin\theta & -e^{-i\tau}\cos\theta \end{pmatrix} \tag{16}$$

where the upper sign corresponds to $\gamma = 0$ and the lower one to $\gamma = \pi$. In connection with this, see Section 5 of ref. 22 and also Lemma 2 in Section 5 of ref. 15, both for the nonrelativistic case (see Appendix C).

Likewise, if the Hamiltonian is invariant under the composed symmetry transformation $\Pi T$,

$$(\Pi T)^{-1} H \Pi T\psi = H\psi \tag{17}$$

then the transformed "spinor" must satisfy $\Pi T\psi \in Dom(H)$. So, the matrix $U$ is such that

$$\sigma_x U^{+} = \overline{U}\sigma_x \tag{18}$$

This requires that

$$a = d \tag{19}$$

In this case two subfamilies of unitary matrices with three parameters are also obtained, one with $\tau = \pi/2$ for the upper sign and other with $\tau = 3\pi/2$ for the lower sign:

$$U = e^{i\mu}\begin{pmatrix} \pm i\cos\theta & e^{i\gamma}\sin\theta \\ e^{-i\gamma}\sin\theta & \pm i\cos\theta \end{pmatrix} \tag{20}$$

Clearly, the set of boundary conditions invariant under the parity transformation $\Pi$ are also invariant under time reversal $T$ and $\Pi T$. Each one of these four subfamilies of boundary conditions can be written as two types of boundary conditions (see Appendix A):

$$\begin{pmatrix} -\chi(0+) \\ \chi(0-) \end{pmatrix} = A_1 \begin{pmatrix} \phi(0+) \\ \phi(0-) \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \phi(0+) \\ \phi(0-) \end{pmatrix} = A_2 \begin{pmatrix} -\chi(0+) \\ \chi(0-) \end{pmatrix}$$

Since $\tau = \pi/2$ or $\tau = 3\pi/2$, there are no boundary conditions associated with Family 3 in appendix A. So we can write the following matrices:

$$A_1 = \frac{i}{\sin\mu - \cos\theta}\begin{pmatrix} \cos\mu & \mp\sin\theta \\ \mp\sin\theta & \cos\mu \end{pmatrix},$$

$$A_2 = \frac{i}{\sin\theta + \cos\theta}\begin{pmatrix} \cos\mu & \pm\sin\theta \\ \pm\sin\theta & \cos\mu \end{pmatrix}$$

where the upper signs correspond to $\gamma = 0$, $\tau = \pi/2$ and the lower ones to $\gamma = \pi$, $\tau = \pi/2$, and

$$A_1 = \frac{i}{\sin\mu + \cos\theta}\begin{pmatrix} \cos\mu & \mp\sin\theta \\ \mp\sin\theta & \cos\mu \end{pmatrix},$$

$$A_2 = \frac{i}{\sin\theta - \cos\theta}\begin{pmatrix} \cos\mu & \pm\sin\theta \\ \pm\sin\theta & \cos\mu \end{pmatrix}$$

where the upper signs correspond to $\gamma = 0$, $\tau = 3\pi/2$ and the lower ones to $\gamma = \pi$, $\tau = 3\pi/2$.

Likewise, in order to recover other self-adjoint extensions already known in the literature, we write each of the subfamilies (11), (12) as two types of subfamilies of boundary conditions (see Appendix B):

$$\begin{pmatrix} \phi(0+) \\ \chi(0+) \end{pmatrix} = B_1\begin{pmatrix} \phi(0-) \\ \chi(0-) \end{pmatrix}, \qquad \begin{pmatrix} \phi(0-) \\ \chi(0-) \end{pmatrix} = B_2\begin{pmatrix} \phi(0+) \\ \chi(0+) \end{pmatrix} \qquad \text{for} \quad \theta \neq 0$$

$$\phi(0+) = -i\,\mathrm{ctn}\left(\frac{\mu + \tau}{2}\right)\chi(0+), \qquad \phi(0-) = -i\tan\left(\frac{\mu - \tau}{2}\right)\chi(0-)$$

$$\text{for} \quad \theta = 0$$

Since the boundary conditions of the first type connect the boundary values of the wavefunction on the left and right half-lines [14], we consider only these types of boundary conditions. So, we can write the following matrices:

$$B_1 = \frac{\pm 1}{\sin\theta}\begin{pmatrix} \cos\mu & i(\sin\mu - \cos\theta) \\ i(\sin\mu + \cos\theta) & \cos\mu \end{pmatrix},$$

$$B_2 = \frac{\pm 1}{\sin\theta}\begin{pmatrix} \cos\mu & -i(\sin\mu - \cos\theta) \\ -i(\sin\mu + \cos\theta) & \cos\mu \end{pmatrix} \qquad (21)$$

where the upper signs correspond to $\gamma = 0$, $\tau = \pi/2$ and the lower ones to $\gamma = \pi$, $\tau = \pi/2$, and

$$B_1 = \frac{\pm 1}{\sin\theta}\begin{pmatrix} \cos\mu & i(\sin\mu + \cos\theta) \\ i(\sin\mu - \cos\theta) & \cos\mu \end{pmatrix},$$
$$B_2 = \frac{\pm 1}{\sin\theta}\begin{pmatrix} \cos\mu & -i(\sin\mu + \cos\theta) \\ -i(\sin\mu - \cos\theta) & \cos\mu \end{pmatrix} \tag{22}$$

where the upper signs correspond to $\gamma = 0$, $\tau = 3\pi/2$ and the lower ones to $\gamma = \pi$, $\tau = 3\pi/2$.

Among the infinite boundary conditions included in $Dom(H)$, with $H$ invariant under space inversion $\Pi$, time reversal $T$, and $\Pi T$, we first obtain two important types of boundary conditions:

1. Those for which the large component $\phi$ is continuous, but the small component $\chi$ is discontinuous at $x = 0$. In the nonrelativistic limit (see Appendix C) these boundary conditions represent a usual delta potential placed at the origin (the wavefunction $\phi_{NR}$ is continuous at $x = 0$, but its first derivative $\phi'_{NR}$ has a jump proportional to the wavefunction at $x = 0$). In other words, all these point interactions describe a nonrelativistic Schrödinger operator perturbed by a nonrelativistic $\delta$ potential: $V(x) = A\delta(x)$. The strength of this potential is a function of only one parameter, $\mu$ or $\theta$. So we can write the following boundary conditions:

$$\begin{pmatrix} \phi(0+) \\ \chi(0+) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 2i \operatorname{ctn}\theta & 1 \end{pmatrix}\begin{pmatrix} \phi(0-) \\ \chi(0-) \end{pmatrix}$$

with $\gamma = 0$, $\tau = \pi/2$, $\mu = \pi/2 - \theta$ or $\gamma = \pi$, $\tau = 3\pi/2$, $\mu = -\pi/2 - \theta$, and

$$\begin{pmatrix} \phi(0+) \\ \chi(0+) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -2i \operatorname{ctn}\theta & 1 \end{pmatrix}\begin{pmatrix} \phi(0-) \\ \chi(0-) \end{pmatrix}$$

with $\gamma = 0$, $\tau = 3\pi/2$, $\mu = -\pi/2 + \theta$ or $\gamma = \pi$, $\tau = \pi/2$, $\mu = \pi/2 + \theta$.

2. An important class of boundary conditions are those for which $\chi$ is continuous and $\phi$ is discontinuous at $x = 0$. In the nonrelativistic limit (see Appendix C) these boundary conditions have $\phi'_{NR}$ continuous and $\phi_{NR}$ discontinuous at the origin [19, 25] and are commonly ill-called [26] $\delta'$ interactions. It is worth noting that these boundary conditions do not describe a nonrelativistic Schrödinger operator perturbed by the derivative of a nonrelativistic $\delta$ potential [27]. Moreover, the nonrelativistic $\delta'$ point interactions so defined are invariant under $x \to -x$, in contrast to $d\delta/dx$, which is an odd function of $x$ [26]. So we can write the following boundary conditions:

$$\begin{pmatrix} \phi(0+) \\ \chi(0+) \end{pmatrix} = \begin{pmatrix} 1 & 2i \operatorname{ctn}\theta \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \phi(0-) \\ \chi(0-) \end{pmatrix}$$

with $\gamma = 0$, $\tau = 3\pi/2$, $\mu = \pi/2 - \theta$ or $\gamma = \pi$, $\tau = \pi/2$, $\mu = -\pi/2 - \theta$, and

$$\begin{pmatrix} \phi(0+) \\ \chi(0+) \end{pmatrix} = \begin{pmatrix} 1 & -2i \operatorname{ctn} \theta \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \phi(0-) \\ \chi(0-) \end{pmatrix}$$

with $\gamma = 0$, $\tau = \pi/2$, $\mu = -\pi/2 + \theta$ or $\gamma = \pi$, $\tau = 3\pi/2$, $\mu = \pi/2 + \theta$.

## 4. GENERALIZED AND USUAL DIRAC DELTA RELATIVISTIC POINT INTERACTIONS

An important third type of boundary conditions included in $Dom(H)$ is obtained making $\theta = \pi/2$ in the matrices $B_1$ and $B_2$ in (21), (22):

$$\begin{pmatrix} \phi(0+) \\ \chi(0+) \end{pmatrix} = \pm \begin{pmatrix} \cos \mu & i \sin \mu \\ i \sin \mu & \cos \mu \end{pmatrix} \begin{pmatrix} \phi(0-) \\ \chi(0-) \end{pmatrix} \tag{23}$$

where $\gamma = 0$, $\pi = \pi/2$ or $3\pi/2$ for the upper sign, and $\gamma = \pi$, $\tau = \pi/2$ or $3\pi/2$ for the lower sign. With this choice ($\theta = \pi/2$) the boundary conditions obtained from (21), (22) are invariant under the replacements $\phi \to \chi$ and $\chi \to \phi$. However, the free Dirac Hamiltonian with point interaction is not invariant under this transformation unless $m = 0$.

All these boundary conditions may be called generalized Dirac delta relativistic point interactions. A particular Dirac delta-type relativistic point interaction is obtained by making in (23) the following replacements: for the upper sign in (23), $\mu \equiv -2 \tan^{-1} (g/2\hbar c)$, where $-\infty < g < 0$ with $0 < \mu < \pi$ and $0 < g < +\infty$ with $\pi < \mu < 2\pi$; for the lower sign in (23), $\mu + \pi \equiv -2 \tan^{-1} (g/2\hbar c)$, where $0 < g < +\infty$ with $0 < \mu < \pi$ and $\mu - \pi \equiv -2 \tan^{-1} (g/2\hbar c)$, where $-\infty < g < 0$ with $\pi < \mu < 2\pi$.

All these boundary conditions correspond to the so-called Dirac $\delta$ relativistic interaction with potential energy $V(x) = g\delta(x)$, and are obtained by directly integrating the Dirac equation [5, 6, 28] making use of $\int_{0-}^{0+} \Psi(x)\delta(x)\, dx = \frac{1}{2}[\Psi(0+) + \Psi(0-)]$ [7]. This relation has been used because, in general, the whole relativistic wavefunction is not continuous at $x = 0$, in contrast with the nonrelativistic case. However, this last relation cannot be imposed in general [7, 29].

In the limit $(g/2\hbar c)^2 \ll 1$, all these boundary conditions [upper or lower sign in (23) with the corresponding $\mu$ replacements] may be rewritten as

$$\begin{pmatrix} \phi(0+) \\ \chi(0+) \end{pmatrix} = \begin{pmatrix} \cos(g/\hbar c) & -i \sin (g/\hbar c) \\ -i \sin(g/\hbar c) & \cos(g/\hbar c) \end{pmatrix} \begin{pmatrix} \phi(0-) \\ \chi(0-) \end{pmatrix} \tag{24}$$

These $g$-dependent boundary conditions correspond to a Dirac delta relativistic interaction with potential energy $V(x) = g\delta(x)$, and may be obtained by solving the Dirac equation for a general sharply peaked potential and then taking the $\delta$-function limit of the potential [7, 30].

The matricial boundary condition (24) seems to be the correct jump condition in the one-dimensional Dirac equation with a local $\delta$-potential. Moreover, it is worth noting that the sign of the strength $g$, positive for repulsive potentials and negative for attractive ones, is not important as far as the existence of bound states is concerned, in accordance with general results for bound states of the one-dimensional Dirac equation [31]. In any case, all these delta boundary conditions are self-adjoint extensions of the Hamiltonian operator perturbed at one point [8, 12].

For very small potential strength, the relativistic boundary conditions (23), with the corresponding $\mu$ replacements, and (24) have the same nonrelativistic limit; in fact, one obtains (see Appendix C)

$$\phi_{NR}(0+) \cong \phi_{NR}(0-) - \frac{g}{2mc^2}\,\phi'_{NR}(0-) \cong \phi_{NR}(0-) \equiv \phi_{NR}(0)$$

$$\phi'_{NR}(0+) - \phi'_{NR}(0-) \cong \frac{2mg}{\hbar^2}\,\phi_{NR}(0)$$

where the primes mean differentiation with respect to $x$. So the Dirac delta relativistic point interactions (23), as well as (24), approach the typical nonrelativistic Dirac delta interaction. For instance, for the "relativistic one-dimensional hydrogen atom" [32] $g = -Ze^2$, and for the upper sign in (23) we obtain $\mu \equiv 2\tan^{-1}(Z\alpha_{fsc}/2)$, where $\alpha_{fsc} \cong 1/137$ is the fine structure constant. For $Z \sim 1$ we can write $\mu \sim \alpha_{fsc}$. In this last case, the results given by the two relativistic deltas (23) and (24) become identical.

## 5. CONCLUSION

In order to relate the various types of boundary conditions included in the domain of the Dirac Hamiltonian appearing in the literature, we have shown that the general four-parameter family of point interactions earlier obtained by Falkensteiner and Grosse may be written in two forms: For one of them three subfamilies of boundary conditions were obtained (see Appendix A). In the nonrelativistic limit one of these subfamilies coincides with those given by Carreau *et al.* and Carreau (see Appendix C). In the other form, three subfamilies of boundary conditions were also obtained, two of which coincide with those studied by Benvegnù and Dabrowski (see Appendix B). In the nonrelativistic limit all of these subfamilies coincide with those studied by Albeverio *et al.* (see Appendix C).

Among the infinite boundary conditions for which the Dirac Hamiltonian operator with point interactions is self-adjoint, we have singled out only those boundary conditions that represent delta-type Dirac point interactions. We first obtain four subfamilies for which the Dirac Hamiltonian perturbed at one point is invariant under space inversion Π. This set of boundary conditions are also invariant under time reversal $T$ and Π$T$. The typical relativistic and nonrelativistic delta-type boundary conditions are, in fact, Π-, $T$-, and Π$T$-invariant. In particular, three important groups of boundary conditions were obtained: One for which the large component $\phi$ is continuous; but the small component $\chi$ is discontinuous at $x = 0$; another for which $\chi$ is continuous, but $\phi$ is discontinuous at $x = 0$; and finally the generalized Dirac delta relativistic point interactions. Obviously, the usual Dirac delta relativistic point interaction is included here.

We believe that our approach relates and unifies in a simple way all previous main results about the several boundary conditions that may be imposed for a free Dirac particle on a line with a hole. Moreover, the basic symmetries that characterize the delta-type Dirac point interactions have been systematically studied.

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## APPENDIX A

The family of boundary conditions included in (4) is

$$\begin{pmatrix} \phi(0+) + \chi(0+) \\ \phi(0-) - \chi(0-) \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \phi(0+) - \chi(0+) \\ \phi(0-) + \chi(0-) \end{pmatrix} \tag{A1}$$

and in order to make contact with the self-adjoint extensions of other authors we write

$$\begin{pmatrix} 1 + a & b \\ c & 1 + d \end{pmatrix} \begin{pmatrix} -\chi(0+) \\ \chi(0-) \end{pmatrix} = \begin{pmatrix} 1 - a & -b \\ -c & 1 - d \end{pmatrix} \begin{pmatrix} \phi(0+) \\ \phi(0-) \end{pmatrix} \tag{A2}$$

Then, three types of families of boundary conditions are obtained:

*Family 1.*

$$\begin{pmatrix} -\chi(0+) \\ \chi(0-) \end{pmatrix} = A_1 \begin{pmatrix} \phi(0+) \\ \phi(0-) \end{pmatrix}, \qquad A_1 = -(A_1)^+ \tag{A3}$$

where the matrix $A_1$ is written as

$$A_1 = i(\sin\mu - \sin\tau\cos\theta)^{-1} \begin{pmatrix} \cos\theta - \cos\tau\cos\theta & -e^{i\gamma}\sin\theta \\ -e^{-i\gamma}\sin\theta & \cos\theta + \cos\tau\cos\theta \end{pmatrix} \tag{A4}$$

with the restriction $\sin\mu - \sin\tau\cos\theta \neq 0$.

*Family 2:*

$$\begin{pmatrix} \phi(0+) \\ \phi(0-) \end{pmatrix} = A_2 \begin{pmatrix} -\chi(0+) \\ \chi(0-) \end{pmatrix}, \qquad A_2 = -(A_2)^+ \tag{A5}$$

where

$$A_2 = i(\sin\mu + \sin\tau\cos\theta)^{-1} \begin{pmatrix} \cos\mu + \cos\tau\cos\theta & e^{i\gamma}\sin\theta \\ e^{-i\gamma}\sin\theta & \cos\mu - \cos\tau\cos\theta \end{pmatrix} \tag{A6}$$

with the restriction $\sin\mu + \sin\tau\cos\theta \neq 0$.

*Family 3.* Finally, let us consider the cases where the above two restrictions are changed to $\sin\mu - \sin\tau\cos\theta = 0$ and $\sin\mu + \sin\tau\cos\theta = 0$. It can be shown that all boundary conditions in this family are obtained from (A2), and are included in some of the following cases:

For $\theta \neq \pi/2$, $0 \leq \gamma < 2\pi$,

$$A_3 \begin{pmatrix} -\chi(0+) \\ \chi(0-) \end{pmatrix} = A_4 \begin{pmatrix} \phi(0+) \\ \phi(0-) \end{pmatrix}, \qquad \mu = 0, \quad \tau = 0 \tag{A7}$$

$$A_5 \begin{pmatrix} -\chi(0+) \\ \chi(0-) \end{pmatrix} = A_6 \begin{pmatrix} \phi(0+) \\ \phi(0-) \end{pmatrix}, \qquad \mu = 0, \quad \tau = \pi \tag{A8}$$

$$A_4 \begin{pmatrix} -\chi(0+) \\ \chi(0-) \end{pmatrix} = A_3 \begin{pmatrix} \phi(0+) \\ \phi(0-) \end{pmatrix}, \qquad \mu = \pi, \quad \tau = 0 \tag{A9}$$

$$A_6 \begin{pmatrix} -\chi(0+) \\ \chi(0-) \end{pmatrix} = A_5 \begin{pmatrix} \phi(0+) \\ \phi(0-) \end{pmatrix}, \qquad \mu = \pi, \quad \tau = \pi \tag{A10}$$

where the matrices $A_3$, $A_4$, $A_5$, and $A_6$ are

$$\begin{gathered} A_3 = \begin{pmatrix} 1 + \cos\theta & e^{i\gamma}\sin\theta \\ e^{-i\gamma}\sin\theta & 1 - \cos\theta \end{pmatrix}, \qquad A_4 = \begin{pmatrix} 1 - \cos\theta & -e^{i\gamma}\sin\theta \\ -e^{-i\gamma}\sin\theta & 1 + \cos\theta \end{pmatrix} \\ A_5 = \begin{pmatrix} 1 - \cos\theta & e^{i\gamma}\sin\theta \\ e^{-i\gamma}\sin\theta & 1 + \cos\theta \end{pmatrix}, \qquad A_6 = \begin{pmatrix} 1 + \cos\theta & -e^{i\gamma}\sin\theta \\ -e^{-i\gamma}\sin\theta & 1 - \cos\theta \end{pmatrix} \end{gathered} \tag{A11}$$

If $\theta = \pi/2$, these boundary conditions are also valid because in this case, from (A2), the matrices $A_3$, $A_4$, $A_5$, and $A_6$ do not depend on $\tau$.

## APPENDIX B

In order to obtain boundary conditions that relate $\psi(0+)$ with $\psi(0-)$, we write (A1) as

$$\begin{pmatrix} 1-a & 1+a \\ c & -c \end{pmatrix}\begin{pmatrix} \phi(0+) \\ \chi(0+) \end{pmatrix} = \begin{pmatrix} b & b \\ 1-d & -1-d \end{pmatrix}\begin{pmatrix} \phi(0-) \\ \chi(0-) \end{pmatrix} \tag{B1}$$

Then, three types of families of boundary conditions may be obtained:

*Family 1:*

$$\begin{pmatrix} \phi(0+) \\ \chi(0+) \end{pmatrix} = B_1\begin{pmatrix} \phi(0-) \\ \chi(0-) \end{pmatrix} \tag{B2}$$

where the matrix $B_1$ is equal to

$$\begin{aligned} B_1 &= e^{i\gamma}(\sin\theta)^{-1}\begin{pmatrix} \cos\mu + \cos\tau\cos\theta & i(\sin\mu - \sin\tau\cos\theta) \\ i(\sin\mu + \sin\tau\cos\theta) & \cos\mu - \cos\tau\cos\theta \end{pmatrix} \\ &\equiv e^{i\gamma}\begin{pmatrix} (B_1)_{11} & (B_1)_{12} \\ (B_1)_{21} & (B_1)_{22} \end{pmatrix} \end{aligned} \tag{B3}$$

with the restriction $\sin\theta \neq 0$. Note that

$$\det\begin{pmatrix} (B_1)_{11} & (B_1)_{12} \\ (B_1)_{21} & (B_1)_{22} \end{pmatrix} = 1$$

so this boundary condition coincides with that given by Benvegnù and Dabrowski [12].

*Family 2:*

$$\begin{pmatrix} \phi(0-) \\ \chi(0-) \end{pmatrix} = B_2\begin{pmatrix} \phi(0+) \\ \chi(0+) \end{pmatrix} \tag{B4}$$

where the matrix $B_2$ is equal to

$$\begin{aligned} B_2 &= e^{-i\gamma}(\sin\theta)^{-1}\begin{pmatrix} \cos\mu - \cos\tau\cos\theta & -i(\sin\mu - \sin\tau\cos\theta) \\ -i(\sin\mu + \sin\tau\cos\theta) & \cos\mu + \cos\tau\cos\theta \end{pmatrix} \\ &\equiv e^{-i\gamma}\begin{pmatrix} (B_2)_{11} & (B_2)_{12} \\ (B_2)_{21} & (B_2)_{22} \end{pmatrix} \end{aligned} \tag{B5}$$

with the same restriction $\sin\theta \neq 0$. This matricial boundary condition is the "inverse" of the condition (B2). In fact, $(B_2)_{11} = (B_1)_{22}$, $(B_2)_{12} = -(B_1)_{12}$, $(B_2)_{21} = -(B_1)_{21}$, and $(B_2)_{22} = -(B_1)_{11}$.

*Family 3:* Finally, we consider the case where $\sin\theta = 0$. This essentially corresponds to the vanishing of the determinants of the square matrices in (B1). So all boundary conditions in this family are obtained by making $\theta = 0$ in (B1),

$$\phi(0+) = -i \operatorname{ctn}\left(\frac{\mu + \tau}{2}\right)\chi(0+), \qquad \phi(0-) = -\, i \tan\left(\frac{\mu - \tau}{2}\right)\chi(0-) \tag{B6}$$

Since $0 \leq \mu, \tau < 2\pi$, then $\operatorname{ctn}(\mu + \tau/2)$ and $\tan(\mu - \tau/2)$ belong to $\Re \cup \{\infty\}$.

It is worth noting that, in addition to (A2), which relates $\{\chi(0+), \chi(0-)\}$ with $\{\phi(0+), \phi(0-)\}$, and (B1), which relates $\{\phi 0+), \chi(0+)\}$ with $\{\phi 0-), \chi(0-)\}$, we can also relate $\{\phi 0+), \chi(0-)\}$ with $\{\phi 0-), \chi(0;+)\}$. Essentially, only these three types of boundary conditions may be obtained, up to $+$, $-$ signs in $\phi$ or $\chi$. This last case is not considered in the literature.

## APPENDIX C

The Dirac equation for stationary states is given by (2). Assuming that the components of the Dirac "spinor" in the Dirac representation satisfy $\phi(x, c) = \phi(x, -c)$, $\chi(x, c) = -\chi(x, -c)$, and $E(c) = E(-c)$, the functions $\phi(x, -c)$ and $\chi(x, -c)$ satisfy equation (2) with $c \to -c$; consequently, we may write the following expansions in $c$ for $\phi(x, c)$ and $\chi(x, c)$:

$$\phi = \phi_{NR} + \frac{1}{c^2}\,\phi_1 + \frac{1}{c_4}\,\phi_2 + \ldots \tag{C1}$$

$$\chi = \frac{1}{c}\,\chi_{NR} + \frac{1}{c^3}\,\chi_1 + \frac{1}{c^4}\,\chi^2 + \ldots$$

and for the energy

$$E = mc^2 + E_{NR} + \frac{1}{c^2}\,E_1 + \frac{1}{c^4}\,E_2 + \ldots \tag{C2}$$

Substituting these expansions in (2) and comparing the terms of the lower order, we obtain

$$i\phi'_{NR} + \frac{2m}{\hbar}\,\chi_{NR} = 0, \qquad i\chi'_{NR} + \frac{E_{NR}}{\hbar}\,\phi_{NR} = 0 \tag{C3}$$

where the primes mean differentiation with respect to $x$. The connection between the components $\phi$ and $\chi$ of the Dirac "spinor" and the Schrödinger eigenfunction $\phi_{NR}$ is obtained by keeping only the first term of the expansions, that is,

$$\phi \to \phi_{NR}, \qquad \chi \to -\lambda i \phi'_{NR} \tag{C4}$$

where $\lambda = \hbar/(2mc)$.

In the nonrelativistic limit the boundary conditions from the first family given in appendix A coincide with those given by Carreau *et al.* [23] and Carreau [24]; in fact, we obtain

$$\begin{pmatrix} -\lambda\phi'_{NR}\,(0+) \\ \lambda\phi'_{NR}\,(0-) \end{pmatrix} = iA_1 \begin{pmatrix} \phi_{NR}\,(0+) \\ \phi_{NR}(0-) \end{pmatrix} \tag{C5}$$

Since the matrix $A_1$ is anti-Hermitian, the matrix $iA_1$ is Hermitian. Likewise, the nonrelativistic limits of the families given in the Appendix B are as follows.

*Family 1:*

$$\begin{pmatrix} \phi_{NR}(0+) \\ \lambda\phi'_{NR}(0+) \end{pmatrix} = e^{i\gamma} \begin{pmatrix} (B_1)_{11} & -i(B_1)_{12} \\ i(B_1)_{21} & (B_1)_{22} \end{pmatrix} \begin{pmatrix} \phi_{NR}(0-) \\ \lambda\phi'_{NR}(0-) \end{pmatrix} \tag{C6}$$

*Family 2:*

$$\begin{pmatrix} \phi_{NR}(0-) \\ \lambda\phi'_{NR}(0-) \end{pmatrix} = e^{-i\gamma} \begin{pmatrix} (B_1)_{22} & i(B_1)_{12} \\ -i(B_1)_{21} & (B_1)_{11} \end{pmatrix} \begin{pmatrix} \phi_{NR}(0+) \\ \lambda\phi'_{NR}(0+) \end{pmatrix} \tag{C7}$$

The $2 \times 2$ matrices in (C6) and (C7) are real, and their determinants are equal to one with $\sin\theta \neq 0$; then these two families can be considered as the same family of boundary conditions.

*Family 3:*

$$\phi_{NR}(0+) = -\mathrm{ctn}\left(\frac{\mu + \tau}{2}\right)\lambda\phi'_{NR}(0+), \tag{C8}$$

$$\phi_{NR}(0-) = -\tan\left(\frac{\mu - \tau}{2}\right)\lambda\phi'_{NR}(0-)$$

Note that in these cases $\sin\theta = 0$. All these families of boundary conditions (Family 1 + 2 and Family 3) are similar to those studied by Kurasov [14] and Albeverio *et al.* [15]. In ref. 14 the boundary conditions of the first and second family are called "connected." Albeverio *et al.* [15] call them "nonseparated." Boundary conditions included in the third family are called by all these authors "separated." In any case, they represent the truly whole family of Schrödinger point interactions, also obtained as the nonrelativistic limit of the general boundary condition included in (4).

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